

ON THE SUM OF THE RECIPROCAL OF CYCLE LENGTHS IN SPARSE GRAPHS

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For a graph G let $\mathcal{L}(G) = \sum \left\{ \frac{1}{k} \mid G \text{ contains a cycle of length } k \right\}$. Erdős and Hajnal [1] introduced the real function $f(\alpha) = \inf \left\{ \mathcal{L}(G) \mid \frac{|E(G)|}{|V(G)|} \cong \alpha \right\}$ and suggested to study its properties. Obviously $f(1) = 0$. We prove $f\left(\frac{k+1}{k}\right) \cong (300k \log k)^{-1}$ for all sufficiently large k , showing that sparse graphs of large girth must contain many cycles of different lengths.

1. Introduction

Let $G = (V, E)$ be a finite undirected graph. Let $\mathcal{L}(G) = \sum \left\{ \frac{1}{k} \mid \text{there exists a cycle of length } k \text{ in } G \right\}$. The number $\mathcal{L}(G)$ is the sum of the reciprocals of cycle lengths occurring in G . In a sense, $\mathcal{L}(G)$ measures how rich the graph G is with respect to cycles. E.g. $\mathcal{L}(K_n) = \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} \approx \left(\gamma - \frac{3}{2} \right) + \log n$, where $\gamma = .5772\dots$ is the Mascheroni constant; $\mathcal{L}(K_{n,n}) = \frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{2n} \approx \frac{1}{2} (\gamma - 1 + \log n)$. Erdős and Hajnal [1] introduced the real function $f(\alpha) = \inf \left\{ \mathcal{L}(G) \mid \frac{|E(G)|}{|V(G)|} \cong \alpha \right\}$. They asked about the behaviour of $f(\alpha)$ as α tends to infinity. The complete bipartite graph $K_{n,n}$ shows that $f(n) \cong c \cdot \log n$ for $n \cong 2$, but originally it was unknown whether $f(\alpha)$ is bounded or not. Recently Gyárfás, Komlós and Szemerédi [2] showed that $f(\alpha) \cong a \cdot \log \alpha$, provided that $\alpha > b$, where a and b are suitable constants. Obviously $f(\alpha) = 0$ for $\alpha \leq 1$. The problem of determining the behaviour of $f(\alpha)$ for $\alpha \in (1, 1 + \varepsilon)$ has been raised in [2]. It was not even clear whether $f(1 + \varepsilon) > 0$ for every $\varepsilon > 0$.

In this paper we prove the following theorem, answering this question in the affirmative:

Theorem 1. *There exists a positive integer n_0 such that $f\left(\frac{k+1}{k}\right) \cong \frac{1}{300k \log k}$ for all integers $k \geq n_0$; in other words: for every graph G with $\frac{|E(G)|}{|V(G)|} \cong \frac{k+1}{k}$ it follows that $\mathcal{L}(G) \cong \frac{1}{300k \log k}$, although the girth of G may be arbitrarily large.*

Basically our proof follows the pattern of [2]: Generalizing the notion of 3/2-tree [2], we construct a subtree of G . Since $\frac{|E(G)|}{|V(G)|} > 1$, eventually we will detect sufficiently many cycles of different lengths, so that $\mathcal{L}(G)$ can be estimated from below.

2. Proof of the theorem

Lemma. *Let k be a positive integer and let G be a graph such that $\frac{|E(G)|}{|V(G)|} \cong \frac{k+1}{k}$. Then there exists a graph G^* such that*

(1) G^* does not contain k vertices x_0, \dots, x_{k-1} , each of which has degree 2 and which form a path of length k .

(2) $\frac{|E(G^*)|}{|V(G^*)|} \cong \frac{k+1}{k}$ and $\mathcal{L}(G^*) \cong \mathcal{L}(G)$.

Proof. We proceed by induction on the number of k -element subsets $\{x_0, \dots, x_{k-1}\}$ of vertices in G such that each x_i has degree 2 and x_0, \dots, x_{k-1} form a path of length k . Pick any such subset $\{x_0, \dots, x_{k-1}\}$ and let \hat{G} be the graph which is obtained from G by deleting $\{x_0, \dots, x_{k-1}\}$ and deleting all edges which are incident with these vertices. Obviously then $\mathcal{L}(\hat{G}) \cong \mathcal{L}(G)$, $|E(\hat{G})| = |E(G)| - k - 1$ and $|V(\hat{G})| = |V(G)| - k$. The assumption $\frac{|E(G)|}{|V(G)|} \cong \frac{k+1}{k}$ implies that also $\frac{|E(\hat{G})|}{|V(\hat{G})|} \cong \frac{k+1}{k}$. Hence, by induction, the assertion follows. ■

Now let $G=(V, E)$ be a fixed finite graph such that

$$|E|/|V| \cong (k+1)/k,$$

for some positive integer k .

According to the lemma we may assume that G is connected and satisfies

(1) G does not contain k vertices x_0, \dots, x_{k-1} , each of which has degree 2 and which form a path in G .

We keep on establishing observations about the interior structure of G . These observations are based on certain reals α, β, γ and g . Ultimately we find out: if these numbers satisfy certain equations (viz. 7.1 and 9.1), then $\mathcal{L}(G)$ is sufficiently large (cf. 17). Since g is directly related to G (cf. 3), it remains to choose α, β and γ accordingly.

The final result is formulated under (20).

- (2) Notation.** A *rooted* ($\cong 2$)-tree in G is a triple $\mathcal{T}=(T, r, F)$ such that
 $T \subseteq V$ is a set of vertices in G ,
 $F \subseteq E$ is a set of edges in G ,
 $r \in T$ is a distinguished vertex, called the *root* of \mathcal{T} ,
 (T, F) is a tree, i.e. a connected graph without cycles, such that every vertex $x \in T$ has degree at most three in (T, F) .

For vertices x and y in T let $\text{dist}(x, y)$ be the number of edges in the (uniquely determined) path in (T, F) joining x and y . For nonnegative integers n let

$$\mathcal{T}(n) = \{x \in T \mid \text{dist}(r, x) = n\}$$

be the n -th level of \mathcal{T} . The *height* of \mathcal{T} is the maximal integer n such that $\mathcal{T}(n) \neq \emptyset$.

- (3) Notation.** Let g be the maximal positive integer such that G does not contain any cycle of length at most g , i.e. $g = \text{girth}(G) - 1$. For a vertex $x \in V$ let $\deg(x)$ denote the number of adjacent vertices, i.e. the degree of x .

- (4) Observation.** Let $0 < \alpha < 1$ be a real number. There exists a nonnegative integer N and there exists a rooted ($\cong 2$)-tree $\mathcal{T}=(T, r, F)$ of height at most $N+1$ satisfying the following properties:

$$(4.1) \quad |\mathcal{T}(l+k)| \cong \alpha^k \frac{2}{2-\alpha^{k-1}} |T(l)| \quad \text{for every } 0 \leq l \leq N-k.$$

Additionally, there exists a set $\text{GOOD} \subseteq \mathcal{T}(N)$ and a mapping $s: \text{GOOD} \rightarrow T$ such that

$$(4.2) \quad |\text{GOOD}| > (1-\alpha) |\mathcal{T}(N)|,$$

- (4.3) $\{x, s(x)\} \in E \setminus F$ for every $x \in \text{GOOD}$, i.e. the vertices x and $s(x)$ are joined by an edge in G not belonging to the rooted tree \mathcal{T} .

Proof. Such a rooted tree \mathcal{T} can be obtained by a straightforward recursive construction. Pick a vertex $r \in V$ arbitrarily and put $T_0 = \{r\}$ and $F_0 = \emptyset$. Assume that for some nonnegative integer m the rooted ($\cong 2$)-tree $\mathcal{T}_m = (T_m, r, F_m)$ has been constructed such that (4.1) is satisfied for all $0 \leq l \leq m-k$.

Let

$$A_m = \{x \in \mathcal{T}_m(m) \mid \deg(x) = 2\}$$

be the set of degree 2 vertices of G belonging to the m -th level of \mathcal{T}_m .

Let

$$A_m^+ = \{y \in V \setminus T_m \mid \{x, y\} \in E \text{ for some } x \in A_m\}$$

be the set of neighbours of A_m not already belonging to \mathcal{T}_m . Let $B_m \subseteq \mathcal{T}_m(m) \setminus A_m$ be a set of maximal cardinality such that there exist two mappings $b_i: B_m \rightarrow V \setminus (T_m \cup A_m^+)$, $i \in \{0, 1\}$, satisfying

$$(4.4) \quad \{x, b_i(x)\} \in E \quad \text{for every } x \in B_m \text{ and } i \in \{0, 1\},$$

$$(4.5) \quad b_i(x) \neq b_j(y) \text{ for every } x, y \in B_m \text{ and } i, j \in \{0, 1\} \text{ with } (x, i) \neq (y, j),$$

i.e. to every $x \in B_m$ there are associated two adjacent vertices $b_0(x)$ and $b_1(x)$ not

already belonging to $T_m \cup A_m^+$ such that $\{b_0(x), b_1(x)\} \cap \{b_0(y), b_1(y)\} = \emptyset$ for any two different x and y in B_m .

Let

$$B_m^+ = \{b_0(x) | x \in B_m\} \cup \{b_1(x) | x \in B_m\}.$$

Put

$$T_{m+1} = T_m \cup A_m^+ \cup B_m^+$$

and

$$F_{m+1} = F_m \cup \{\{x, y\} \in E | x \in A_m \text{ and } y \in A_m^+\} \cup \{\{x, b_0(x)\} | x \in B_m\} \cup \{\{x, b_1(x)\} | x \in B_m\}.$$

Repeat this construction as long as possible, viz. let N be the minimal integer such that

$$(4.6) \quad |A_N| + |B_N| < \alpha |\mathcal{T}_N(N)|,$$

and put $\mathcal{T} = (T, r, F) = (T_{N+1}, r, F_{N+1})$.

Recall that \mathcal{T} has been defined in such a way that

$$(4.7) \quad |A_m| + |B_m| \cong \alpha |\mathcal{T}(m)| \quad \text{for every } 0 \leq m < N$$

and

$$(4.8) \quad |\mathcal{T}(m)| = |A_{m-1}| + 2|B_{m-1}| \quad \text{for every } 0 < m \leq N.$$

We show that \mathcal{T} satisfies assertion (4.1): Let $0 \leq l \leq N-k$. By (4.7) then

$$(4.9) \quad \alpha^{k-1} |A_l| + \alpha^{k-1} |B_l| \cong \alpha^k |T(l)|.$$

According to (4.7) and (4.8) it follows that

$$(4.10) \quad \alpha^{k-1-i} |A_{l+i}| + \alpha^{k-1-i} |B_{l+i}| - \alpha^{k-i} |A_{l-1+i}| - 2\alpha^{k-i} |B_{l-1+i}| \cong 0$$

for every $1 \leq i \leq k$.

Summing up the inequalities (4.9) and (4.10) yields

$$(4.11) \quad |A_{l+k-1}| + |B_{l+k-1}| - \sum_{i=0}^{k-2} \alpha^{k-1-i} |B_{l+i}| \cong \alpha^k |\mathcal{T}(l)|.$$

As the graph satisfies condition (1), it follows that

$$(4.12) \quad |A_{l+k-1}| \leq \sum_{i=0}^{k-2} 2|B_{l+i}|,$$

and hence

$$(4.13) \quad -\frac{\alpha^{k-1}}{2} |A_{l+k-1}| + \sum_{i=0}^{k-2} \alpha^{k-1-i} |B_{l+i}| \cong 0.$$

Summing up (4.11) and (4.13) yields

$$(4.14) \quad \frac{2-\alpha^{k-1}}{2} |A_{l+k-1}| + |B_{l+k-1}| + \sum_{i=1}^{k-2} (\alpha^{k-1} - \alpha^{k-1-i}) |B_{l+i}| \cong \alpha^k |\mathcal{T}(l)|,$$

thus

(4.15)

$$\begin{aligned} |A_{l+k-1}| &\cong \frac{2\alpha^k}{2-\alpha^{k-1}} |\mathcal{T}(l)| + \frac{2}{\alpha^{k-1}-2} (|B_{l+k-1}| + \sum_{i=1}^{k-2} (\alpha^{k-1} - \alpha^{k-1-i}) |B_{l+i}|) \\ &\cong \frac{2\alpha^k}{2-\alpha^{k-1}} |\mathcal{T}(l)| + \frac{2}{\alpha^{k-1}-2} |B_{l+k-1}|, \end{aligned}$$

as $0 < \alpha < 1$.

From (4.8) and (4.15) then finally it follows that

(4.16)

$$\begin{aligned} |\mathcal{T}(l+k)| &= \\ &= |A_{l+k-1}| + 2|B_{l+k-1}| \cong \frac{2\alpha^k}{2-\alpha^{k-1}} |\mathcal{T}(l)| + \frac{2(\alpha^{k-1}-1)}{\alpha^{k-1}-2} |B_{l+k-1}| \cong \frac{2\alpha^k}{2-\alpha^{k-1}} |\mathcal{T}(l)|, \end{aligned}$$

as again $0 < \alpha < 1$. Thus assertion (4.1) is valid.

Let us mention that in fact the sets A_{l+i} and B_{l+i} can be chosen in such a way that equality holds in (4.1) occasionally.

It remains to define the set GOOD and the mapping $s: \text{GOOD} \rightarrow T$ such that (4.2) and (4.3) are satisfied. Put

$$\text{GOOD} = \mathcal{T}(N) \setminus (A_N \cup B_N).$$

According to (4.6) then (4.2) is valid. Due to the maximality of B_N for every $x \in \text{GOOD}$ there exists some vertex $s(x) \in T = T_{N+1}$ satisfying (4.3). ■

For the remainder of this proof fix a rooted (≥ 2) -tree $\mathcal{T} = (T, r, F)$ of height N or $N+1$ and a set $\text{GOOD} \subseteq \mathcal{T}(N)$ as well as a mapping $s: \text{GOOD} \rightarrow T$ such that the assertions (4.1), (4.2) and (4.3) are satisfied.

(5) Observation. *There exists a set $\text{EQP} \subseteq \text{GOOD}$ and there exists an integer $n \cong g$ such that*

$$(5.1) \quad \text{dist}(x, s(x)) = n \quad \text{for every } x \in \text{EQP},$$

$$(5.2) \quad |\text{EQP}| \cong |\text{GOOD}|/n^2,$$

$$(5.3) \quad |\text{EQP}| \cong \frac{1-\alpha}{n^2} \left(\alpha \sqrt{\frac{2}{2-\alpha^{k-1}}} \right)^{N-k+1}.$$

Proof. Recall that $g = \text{girth}(G) - 1$. Consider the mapping $\Delta: \text{GOOD} \rightarrow \mathbb{N} \setminus \{0, \dots, g-1\}$ which is defined by $\Delta(x) = \text{dist}(x, s(x))$ for every $x \in \text{GOOD}$. Then $|\text{GOOD}| = \sum_{n=g}^{\infty} |\Delta^{-1}(n)|$ and assuming that (5.2) fails for every $n \cong g$ and $\text{EQP} = \Delta^{-1}(n)$ it follows that

$$|\text{GOOD}| = \sum_{n=g}^{\infty} |\Delta^{-1}(n)| < |\text{GOOD}| \sum_{n=2}^{\infty} \frac{1}{n^2} = |\text{GOOD}|(\pi^2/6 - 1) < |\text{GOOD}|,$$

an obvious contradiction.

Hence for some $n \geq g$ the set $A^{-1}(n)$ satisfies (5.1) and (5.2). Assertion (5.3) is an obvious consequence from (4.2) and (4.1) and (5.2). ■

Fix an integer $n \geq g$ and a set $\text{EQP} \subseteq \text{GOOD}$ satisfying (5.1), (5.2) and (5.3). Put $n^* = \lfloor n/2 - 1 \rfloor$.

(6) Notation. For vertices $z \in T$ let

$$C(z) = \{y \in \mathcal{T}(N) \mid \text{dist}(z, y) = N - \text{dist}(r, y)\}$$

be the vertices in the N -th level of \mathcal{T} belonging to the cone generated by z .

(7) Observation. Let $0 < \beta < 1$ be a real number such that

$$(7.1) \quad \frac{1-\alpha}{2g^2} \geq \beta^{(g-1)/2}.$$

There exists a vertex $z \in \mathcal{T}(N-n^*)$ such that

$$(7.2) \quad |C(z) \cap \text{EQP}| \geq \frac{1-\alpha}{2n^2} \beta^{n^*} \left(\alpha \sqrt{\frac{2}{2-\alpha^{k-1}}} \right)^{n^*-k+1}.$$

Proof. Call a vertex $z \in \mathcal{T}(N-n^*)$ *small* iff

$$(7.3) \quad |C(z)| < \beta^{n^*} \left(\alpha \sqrt{\frac{2}{2-\alpha^{k-1}}} \right)^{n^*-k+1}.$$

Without loss of generality we can assume that $|C(z)| \neq 0$ for every $z \in \mathcal{T}(N-n^*)$. Hence it follows from (4.1) that

$$(7.4) \quad \begin{aligned} \left| \bigcup_{\substack{z \in \mathcal{T}(N-n^*) \\ z \text{ small}}} C(z) \right| &< |\mathcal{T}(N-n^*)| \beta^{n^*} \left(\alpha \sqrt{\frac{2}{2-\alpha^{k-1}}} \right)^{n^*-k+1} \\ &\leq |\mathcal{T}(N)| \left(\alpha \sqrt{\frac{2}{2-\alpha^{k-1}}} \right)^{k-1-n^*} \beta^{n^*} \left(\alpha \sqrt{\frac{2}{2-\alpha^{k-1}}} \right)^{n^*-k+1} \\ &= |\mathcal{T}(N)| \beta^{n^*}. \end{aligned}$$

Note that particularly $g \geq 2$ (since $\text{girth}(G)$ is at least 3). From $n \geq g \geq 2$ and (7.1) it follows that

$$(7.5) \quad \frac{1-\alpha}{2n^2} \geq \beta^{n^*}.$$

Denote by EQP^* the set of vertices in EQP which are not covered by small cones. According to (7.4), (5.2), (4.2), (7.1) and since $n \geq g$ it follows that

$$(7.6) \quad \begin{aligned} |\text{EQP}^*| &\geq |\text{EQP}| - \left| \bigcup_{\substack{z \in \mathcal{T}(N-n^*) \\ z \text{ small}}} C(z) \right| \geq |\text{EQP}| - |\mathcal{T}(N)| \beta^{n^*} \\ &\geq |\mathcal{T}(N)| \frac{1-\alpha}{n^2} - |\mathcal{T}(N)| \beta^{n^*} = |\mathcal{T}(N)| \left(\frac{1-\alpha}{n^2} - \beta^{n^*} \right) \\ &\geq |\mathcal{T}(N)| \frac{1-\alpha}{2n^2}. \end{aligned}$$

Note that $C(z) \cap C(z') = \emptyset$ for different vertices z and z' in $\mathcal{T}(N-n^*)$. Hence there exist at most

$$|\mathcal{T}(N)|/\beta^{n^*} \left(\alpha \sqrt[k]{\frac{2}{2-\alpha^{k-1}}} \right)^{n^*-k+1}$$

vertices $z \in \mathcal{T}(N-n^*)$ which are not small. Assuming that every such "large" vertex fails to satisfy (7.2) yields

$$|\text{EQP}^*| < \frac{1-\alpha}{2n^2} |\mathcal{T}(N)|,$$

contradicting (7.6). Hence the observation is proved. ■

Fix a vertex $z \in \mathcal{T}(N-n^*)$ satisfying (7.2).

(8) Notation. Let $m < N$. A vertex $x \in \mathcal{T}(m)$ is a *cycle vertex* iff there exist two vertices y_0 and y_1 in $\mathcal{T}(m+1)$ such that $\{x, y_0\} \in F$ and $\{x, y_1\} \in F$ (i.e. x possesses two immediate successors in \mathcal{T} , viz. y_0 and y_1) satisfying

$$(8.1) \quad C(y_0) \cap \text{EQP} \neq \emptyset \quad \text{and} \quad C(y_1) \cap \text{EQP} \neq \emptyset.$$

(9) Observation. Let $\gamma > 0$ be a real number and assume that α, β, g (as introduced above) and γ satisfy

$$(9.1) \quad \log \left(\frac{1-\alpha}{2n^2} \right) + \left(\frac{n}{2} - 1 \right) \log \beta + \left(\frac{n}{2} - k - 1 \right) \log \left(\alpha \sqrt[k]{\frac{2}{2-\alpha^{k-1}}} \right) \geq n\gamma \log 2$$

for all $n \geq g$. Then there exists an ascending path $z = x_0, x_1, \dots, x_{n^*-1}$ in \mathcal{T} (i.e. $\{x_i, x_{i+1}\} \in F$ and $x_i \in \mathcal{T}(N-n^*+i)$) which contains at least γn cycle vertices.

Proof. Construct x_1, \dots, x_{n^*-1} by a greedy procedure. If there exists precisely one vertex $y \in \mathcal{T}(N-n^*+m+1)$ such that $\{x_m, y\} \in F$, put $x_{m+1} = y$. If there exist two vertices y_0 and y_1 in $\mathcal{T}(N-n^*+m+1)$ such that $\{x_m, y_0\} \in F$ and $\{x_m, y_1\} \in F$, put

$$\begin{aligned} x_{m+1} = y_0 & \quad \text{iff} \quad |C(y_0) \cap \text{EQP}| \geq |C(y_1) \cap \text{EQP}| \\ & = y_1 \quad \text{iff} \quad |C(y_0) \cap \text{EQP}| < |C(y_1) \cap \text{EQP}|. \end{aligned}$$

We show that $z = x_0, \dots, x_{n^*-1}$ contains at least γn cycle vertices. Let $x_{m_0}, \dots, x_{m_{t-1}}$, where $m_0 < m_1 < \dots < m_{t-1}$, be the cycle vertices amongst x_0, \dots, x_{n^*-1} .

According to the construction and (8.1) it follows that

$$(9.2) \quad |C(x_{m_{t-1}}) \cap \text{EQP}| \leq 2,$$

$$(9.3) \quad |C(x_{m_{l-1}}) \cap \text{EQP}| \leq 2|C(x_{m_l}) \cap \text{EQP}| \quad \text{for} \quad 0 < l < t.$$

Hence

$$(9.4) \quad |C(x_{m_0}) \cap \text{EQP}| \leq 2^t.$$

Note that $|C(x_{m_0}) \cap \text{EQP}| = |C(z) \cap \text{EQP}|$, hence it follows from (7.2) that

$$(9.5) \quad 2^t \leq \frac{1-\alpha}{2n^2} \beta^{n^*} \left(\alpha \sqrt[k]{\frac{2}{2-\alpha^{k-1}}} \right)^{n^*-k+1}$$

i. e.

$$\begin{aligned}
 (9.6) \quad t &\cong \frac{1}{\log 2} \left(\log \left(\frac{1-\alpha}{2n^2} \right) + n^* \log \beta + (n^* - k + 1) \log \left(\alpha \sqrt[k]{\frac{2}{2-\alpha^{k-1}}} \right) \right) \\
 &\cong \frac{1}{\log 2} \left(\log \left(\frac{1-\alpha}{2n^2} \right) + \left(\frac{n}{2} - 1 \right) \log \beta + \left(\frac{n}{2} - k - 1 \right) \log \left(\alpha \sqrt[k]{\frac{2}{2-\alpha^{k-1}}} \right) \right) \\
 &\cong \gamma n
 \end{aligned}$$

according to (9.1) and the fact that $n \cong g$. ■

Fix a path $x_0, x_1, \dots, x_{n^*-1}$ containing the cycle vertices $x_{m_0}, \dots, x_{m_{t-1}}$, where $m_0 < m_1 < \dots < m_{t-1}$ and $t \cong \gamma n$. Fix a vertex $x_{m_t} \in C(x_{m_{t-1}}) \cap \text{EQP}$. For every vertex x_{m_i} let $y_{m_i} \in \text{EQP}$ be such that

$$(10) \quad y_{m_i} \in C(x_{m_i}) \cap \text{EQP} \setminus C(x_{m_{i+1}}) \text{ for every } 0 \leq i < t.$$

Put $y_{m_t} = x_{m_t}$. Such vertices y_{m_i} exist as the vertices x_{m_i} are cycle vertices.

(11) Notation. For vertices x and y in T denote by $P(x, y)$ the (uniquely determined) path in \mathcal{T} joining x and y . In order to avoid ambiguities say that $P(x, y)$ consists of edges (in F), determining the path between x and y . Note that $|P(x, y)| = \text{dist}(x, y)$.

(12) Observation. Let $z^* \in T$ denote the vertex satisfying $\{z^*, z\} \in P(r, z)$, i.e. z^* is the (uniquely determined) predecessor of z in T . Then $\{z^*, z\} \in P(y_{m_i}, s(y_{m_i}))$ for every $0 \leq i \leq t$.

Proof. Denote by $\bar{C}(z) = \bigcup_{0 \leq i \leq n^*+1} \{y \in \mathcal{T}(N - n^* + i) \mid \text{dist}(y, z) = i\}$ the (upper) cone generated by z . Obviously

$$(12.1) \quad \text{MAX} \{ \text{dist}(y, z) \mid y \in \bar{C}(z) \} \leq n^* + 1,$$

hence

$$(12.2) \quad \text{MAX} \{ \text{dist}(x, y) \mid y \in C(z), x \in \bar{C}(z) \} \leq n^* + n^* + 1 = 2 \left\lfloor \frac{n}{2} - 1 \right\rfloor + 1 \leq n - 1.$$

From (5.1) it follows that $s(y_{m_i}) \notin \bar{C}(z)$ for any $0 \leq i \leq t$. Hence the assertion follows. ■

(13) Observation. Let $0 \leq i < j \leq t$. Then

$$(13.1) \quad P(y_{m_i}, y_{m_j}) \cap P(s(y_{m_i}), s(y_{m_j})) = \emptyset.$$

$$(13.2) \quad C_{i,j} = P(y_{m_i}, y_{m_j}) \cup P(s(y_{m_i}), s(y_{m_j})) \cup \{y_{m_i}, s(y_{m_i})\}, \{y_{m_j}, s(y_{m_j})\}$$

forms a cycle of length at most $2n$.

Proof. As the only way of entering or leaving the cone $\bar{C}(z)$ is to use the edge $\{z, z^*\}$, assertion (13.1) follows from (12). Consequently C_{ij} is a cycle. As $y_{m_i} \in C(x_{m_i}) \setminus C(x_{m_{i+1}})$ and $\{x_{m_i}, x_{m_{i+1}}\} \in F$ it follows from $i < j$ that

$$(13.3) \quad \text{dist}(y_{m_i}, y_{m_j}) = 2(n^* - m_i).$$

From (12) and (5.1) it follows that

$$(13.4) \quad \text{dist}(z^*, s(y_{m_i})) = \text{dist}(y_{m_i}, s(y_{m_i})) - \text{dist}(y_{m_i}, z^*) = n - n^* - 1,$$

hence by the triangle inequality

$$(13.5) \quad \text{dist}(s(y_{m_i}), s(y_{m_j})) \leq 2(n - n^* - 1),$$

this yields

$$(13.6) \quad |P(y_{m_i}, y_{m_j}) \cup P(s(y_{m_i}), s(y_{m_j})) \cup \{\{y_{m_i}, s(y_{m_i})\}, \{y_{m_j}, s(y_{m_j})\}\}| \\ \leq 2(n^* - m_i) + 2(n - n^* - 1) + 2 = 2n - 2m_i \leq 2n$$

showing that C_{ij} is a cycle of length at most $2n$. ■

(14) Fact. Let $\mathcal{T}^* = (T^*, r^*, F^*)$ be a rooted tree with distance function dist^* . For every nonnegative integer m and every vertex $\hat{x} \in \mathcal{T}^*(m)$ it follows that

$$(14.1) \quad \text{MAX} \{\text{dist}^*(\hat{x}, y) | y \in \mathcal{T}^*(m)\} = \text{MAX} \{\text{dist}^*(y, y') | y, y' \in \mathcal{T}^*(m)\}.$$

Proof. Obvious by induction on m . ■

(15) Observation. For nonnegative integers $0 \leq i < t$ let $i < \xi(i) \leq t$ be such that

$$(15.1) \quad \text{dist}(s(y_{m_i}), s(y_{m_{\xi(i)}})) = \text{MAX} \{\text{dist}(s(y_{m_i}), s(y_{m_j})) | i < j \leq t\}.$$

Then

$$(15.2) \quad \text{dist}(s(y_{m_{i+1}}), s(y_{m_{\xi(i+1)}})) \leq \text{dist}(s(y_{m_i}), s(y_{m_{\xi(i)}})).$$

Proof. Consider

$$T^* = \{y \in T | \text{dist}(z^*, y) \leq n - n^* - 1\} \setminus \bar{C}(z)$$

and for every $0 \leq i < t$ let

$$T_i^* = T^* \cup \{s(y_{m_j}) | i \leq j \leq t\}.$$

Denote by \mathcal{T}_i^* the rooted subtree of \mathcal{T} consisting of vertices T_i^* with root z^* . According to (12) and (13.4) then

$$(15.3) \quad \mathcal{T}_i^*(n - n^* - 1) = \{s(y_{m_j}) | i \leq j \leq t\}.$$

As $T_i^* \supseteq T_{i+1}^*$ it follows that

$$(15.4) \quad \text{MAX} \{\text{dist}_{\mathcal{T}_{i+1}^*}(y, y') | y, y' \in \mathcal{T}_{i+1}^*(n - n^* - 1)\} \\ \leq \text{MAX} \{\text{dist}_{\mathcal{T}_i^*}(y, y') | y, y' \in \mathcal{T}_i^*(n - n^* - 1)\}.$$

According to (14.1) then

$$(15.5) \quad \text{dist}(s(y_{m_{i+1}}), s(y_{m_{\xi(i+1)}})) = \text{dist}_{\mathcal{T}_{i+1}^*}(s(y_{m_{i+1}}), s(y_{m_{\xi(i+1)}})) \\ \leq \text{dist}_{\mathcal{T}_i^*}(s(y_{m_i}), s(y_{m_{\xi(i)}})) = \text{dist}(s(y_{m_i}), s(y_{m_{\xi(i)}})). \quad \blacksquare$$

(16) Observation. $|C_{i+1, \xi(i+1)}| < |C_{i, \xi(i)}|$ for every $0 \leq i < t$.

Proof. According to (13.3) and since $m_i < m_{i+1}$ this follows from (15.2). ■

(17) Observation. Let the real numbers $0 < \alpha < 1$, $0 < \beta < 1$ and $\gamma > 0$ satisfy (7.1) and (9.1). Then

$$\mathcal{L}(G) \cong \gamma/2. \quad \blacksquare$$

(18) Observation. Put $\alpha = 2^{-1/2k}$; $\beta = \left(\frac{2}{2\sqrt{2} - \sqrt[2k]{2}}\right)^{-1/2k}$; $\gamma = 0.0011/k$. Then (9.1) is satisfied, provided that $g > 300k \log k$, and $k \geq n_0$, where n_0 is sufficiently large.

Proof. We write (9.1) for $n=g$ equivalently as

$$(18.1) \quad \log\left(\frac{1-\alpha}{2g^2}\right) + (g/2 - 1) \log\left(\alpha\beta \sqrt[k]{\frac{2}{2-\alpha^{k-1}}}\right) - k \log\left(\alpha \sqrt[k]{\frac{2}{2-\alpha^{k-1}}}\right) \cong g\gamma \log 2$$

and evaluate each of the three summands on the left hand side of (18.1) separately.

$$(18.2) \quad \log\frac{1-\alpha}{2g^2} = \log(1 - 2^{-1/2k}) - \log 2 - 2 \log g \cong \log\left(\frac{\log 2}{2k}\right) - \log 2 - 2 \log g \\ \cong -(2 \log 2 + \log k + 2 \log g).$$

$$(18.3) \quad \alpha \sqrt[k]{\frac{2}{2-\alpha^{k-1}}} = \sqrt[k]{\frac{2}{2\sqrt[2k]{2} - \sqrt[2k]{2}}}.$$

$$(18.4) \quad \alpha\beta \sqrt[k]{\frac{2}{2-\alpha^{k-1}}} = \sqrt[2k]{\frac{2}{2\sqrt[2k]{2} - \sqrt[2k]{2}}},$$

$$(18.5) \quad (g/2 - 1) \log\left(\alpha\beta \sqrt[k]{\frac{2}{2-\alpha^{k-1}}}\right) = (g/2 - 1) \log\left(\sqrt[2k]{\frac{2}{2\sqrt[2k]{2} - \sqrt[2k]{2}}}\right) \\ = \frac{1}{2k} (g/2 - 1) \log\left(\frac{2}{2\sqrt[2k]{2} - \sqrt[2k]{2}}\right) \\ \cong \frac{g}{4k} 0.088 - \frac{1}{2k} 0.088 \\ = g0.022/k - \frac{0.044}{k},$$

$$(18.6) \quad -k \log\left(\alpha \sqrt[k]{\frac{2}{2-\alpha^{k-1}}}\right) \cong \log(2\sqrt{2} - 1) - \log 2 \cong -\log 2.$$

Now, putting (18.2), (18.5), and (18.6) together shows that

$$\log\left(\frac{1-\alpha}{2g^2}\right) + (g/2 - 1) \log\left(\alpha\beta \sqrt[k]{\frac{2}{2-\alpha^{k-1}}}\right) - k \log\left(\alpha \sqrt[k]{\frac{2}{2-\alpha^{k-1}}}\right) \\ \cong g0.022/k - \left(\frac{0.044}{k} + 3 \log 2 + \log k\right) - 2 \log g \\ = g0.011/k + g \cdot 0.011/k - \left(\frac{0.044}{k} + 3 \log 2 + \log k\right) - 2 \log g$$

and thus it suffices to show that

$$(18.7) \quad g \cdot 0.011/k - \left(\frac{0.044}{k} + 3 \log 2 + \log k \right) - 2 \log g \cong 0.$$

This follows immediately according to the choices of g and k . To show (9.1) for arbitrary $n \cong g$ it is enough to see

$$(18.8) \quad \frac{1}{2} \log \beta + \frac{1}{2} \log \left(\alpha \sqrt[k]{\frac{2}{2 - \alpha^{k-1}}} \right) - \gamma \log 2 - 2 \log \left(\frac{g+1}{g} \right) \cong 0$$

but this also holds by the choices of α, β, γ, g and k . ■

(19) Observation. Put α and β as before, then (7.1) is satisfied, provided $g \cong 300k \log k$ and $k \cong n_0$ for some sufficiently large n_0 .

Proof.

$$(19.1) \quad \log(\beta^{g-4/2}) = \frac{g-4}{2} \frac{(-1)}{2k} \log \frac{2}{2\sqrt[2k]{2} - \sqrt[2k]{2}} \cong \frac{g-4}{2} \frac{(-1)}{2k} \log 2$$

thus, according to (18.2) and (19.1), it suffices to show that

$$\frac{g-4}{4k} \log 2 \cong (2 \log 2 + \log k + 2 \log g),$$

this follows immediately, as $g \cong 300k \log k$ and $k \cong n_0$. ■

(20) Observation. There exists a positive integer n_0 such that for every graph $G = (V, E)$ with $\frac{|E|}{|V|} \cong \frac{k+1}{k}$, where $k \cong n_0$, it follows that $\mathcal{L}(G) \cong \min \left\{ \frac{1}{300k \log k}, 0.00055/k \right\}$.

Proof. Recall (17), (18) and (19). ■

3. Concluding Remarks

We have the following upper bound for $f(1+\varepsilon)$:

Theorem 2. $f\left(\frac{k+1}{k}\right) \cong \frac{1}{k+1} \frac{77}{72}$ for all positive integers $k \equiv 2 \pmod{15}$.

Proof. For the positive integers l and n denote by $K_{n,n}^l$ the graph resulting from the complete bipartite graph $K_{n,n}$ by inserting l additional vertices on each edge of $K_{n,n}$. Clearly

$$|V(K_{n,n}^l)| = 2n + n^2 l, |E(K_{n,n}^l)| = n^2(l+1),$$

$$\frac{|E(K_{n,n}^l)|}{|V(K_{n,n}^l)|} = \frac{l+1}{l + \frac{2}{n}}, \quad \mathcal{L}(K_{n,n}^l) = \frac{1}{l+1} \left(\frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \dots + \frac{1}{2n} \right).$$

Putting $f = \frac{k(n-2)-2}{n}$ shows that $f\left(\frac{k+1}{k}\right) \cong \frac{1}{k+1} \frac{n}{n-2} \left(\frac{1}{4} + \frac{1}{6} + \dots + \frac{1}{2n}\right)$ and this expression is minimized for $n=5$, hence $f\left(\frac{k+1}{k}\right) \cong \frac{1}{k+1} \frac{77}{72}$, provided $k \equiv 2 \pmod{15}$. ■

A careful inspection of the proof of our main theorem suggests the following definition:

Definition. For positive reals α and g let

$$f^*(g, \alpha) = \inf \left\{ \mathcal{L}(G) \mid \frac{|E(G)|}{|V(G)|} \cong \alpha \text{ and } \text{girth}(G) \cong g \right\}.$$

Corollary. There exists a positive integer n_0 such that

$$f^*\left(300k \log k, \frac{k+1}{k}\right) \cong \frac{0.00055}{k}$$

for all positive integers $k \cong n_0$.

Problem. Fix $\alpha > 1$, is $f^*(g, \alpha)$ bounded as g tends to infinity?

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